

$SL(2, \mathbb{R})$ -INVARIANT PROBABILITY MEASURES ON THE MODULI SPACES OF TRANSLATION SURFACES ARE REGULAR

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ABSTRACT. In the moduli space \mathcal{H}_g of normalized translation surfaces of genus g , consider, for a small parameter $\rho > 0$, those translation surfaces which have two non-parallel saddle-connections of length $\leq \rho$. We prove that this subset of \mathcal{H}_g has measure $o(\rho^2)$ w.r.t. any probability measure on \mathcal{H}_g which is invariant under the natural action of $SL(2, \mathbb{R})$. This implies that any such probability measure is regular, a property which is important in relation with the recent fundamental work of Eskin-Kontsevich-Zorich on the Lyapunov exponents of the KZ-cocycle.

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1. INTRODUCTION

1.1. Regular $SL(2, \mathbb{R})$ -invariant probability measures on \mathcal{H}_g . The study of interval exchange transformations and translation flows on translation surfaces (such as billiards in rational polygons) was substantially advanced in the last 30 years because of its intimate

relationship with the moduli spaces \mathcal{H}_g of normalized (unit area) Abelian differentials on compact Riemann surfaces of genus $g \geq 1$. In fact, the moduli spaces of normalized Abelian differentials admit a natural action of $SL(2, \mathbb{R})$ and this action works as a renormalization dynamics for interval exchange transformations and translation flows. In particular, this fundamental observation was successfully applied by various authors to derive several remarkable results:

- in the seminal works of H. Masur [12] and W. Veech [18] in the early 80's, the recurrence properties of the orbits of the diagonal subgroup $g_t = \text{diag}(e^t, e^{-t})$ of $SL(2, \mathbb{R})$ on \mathcal{H}_g were a key tool in the solution of Keane's conjecture on the unique ergodicity of typical interval exchange transformations;
- after the works of A. Zorich [22], [23] in the late 90's and G. Forni [8] in 2001, we know that the Lyapunov exponents of the action of the diagonal subgroup $g_t = \text{diag}(e^t, e^{-t})$ of $SL(2, \mathbb{R})$ on \mathcal{H}_g drive the deviations of ergodic averages of typical interval exchange transformations and translation flows;
- more recently, V. Delecroix, P. Hubert and S. Lelièvre [3] confirmed a conjecture of the physicists J. Hardy and J. Weber on the abnormal rate of diffusion of typical trajectories in typical realizations of the Ehrenfest wind-tree model of Lorenz gases by relating this question to certain Lyapunov exponents of the diagonal subgroup $g_t = \text{diag}(e^t, e^{-t})$ of $SL(2, \mathbb{R})$ on \mathcal{H}_5 .

It is worth to point out that the applications in the last two items above involved Lyapunov exponents of the diagonal subgroup $g_t = \text{diag}(e^t, e^{-t})$ of $SL(2, \mathbb{R})$ on the moduli spaces of normalized Abelian differentials, and this partly explains the recent literature dedicated to this subject (e.g., [1], [2], [4], [9] and [15]).

In this direction, the most striking recent result is arguably the formula of A. Eskin, M. Kontsevich and A. Zorich [5] relating sums of Lyapunov exponents to certain geometrical quantities called Siegel-Veech constants. Very roughly speaking, the derivation of this formula can be described as follows (see also the excellent Bourbaki seminar [10] by Grivaux-Hubert on this subject). Given m an ergodic $SL(2, \mathbb{R})$ -invariant probability measure on \mathcal{H}_g , a remarkable formula of M. Kontsevich [11] and G. Forni [8] allows to express the sum of the top g Lyapunov exponents of m in terms of the curvature of the determinant line bundle of the Hodge bundle over the support of m . The formula of Kontsevich and Forni suffices for some particular examples of m , but, in general, its range of applicability is limited because it is not easy to compute the curvature of the determinant line bundle of the Hodge bundle. For this reason, A. Eskin, M. Kontsevich and A. Zorich use an analytic version of the Riemann-Roch-Hirzebruch-Grothendieck theorem to express the curvature of the determinant of the Hodge bundle as the sum of a simple combinatorial term $\frac{1}{12} \sum_{i=1}^{\sigma} \frac{k_i(k_i+2)}{k_i+1}$ depending only on the orders k_1, \dots, k_{σ} of the zeroes of the Abelian differentials in the support of m and an integral expression I depending on the flat geometry of the Abelian differentials in the support of m . At this point, A. Eskin, M. Kontsevich and A. Zorich complete the derivation of their formula with an integration by parts argument in order to relate the integral expression I mentioned above to Siegel-Veech constants associated to the problem of counting maximal (flat) cylinders of the Abelian differentials in the support of m .

Concerning the hypothesis for the validity of the formula of A. Eskin, M. Kontsevich and A. Zorich, as the authors point out in their article [5], most of their arguments use only the $SL(2, \mathbb{R})$ -invariance of the ergodic probability measure m . Indeed, the sole place

where they need an extra assumption on m is precisely in Section 9 of [5] for the justification of the integration by parts argument mentioned above (relating a certain integral expression I to Siegel-Veech constants).

Concretely, this extra assumption is called *regularity* in [5] and it is defined as follows (see Subsections 1.5 and 1.6 of [5]). Given a normalized Abelian differential ω on a Riemann surface M of genus $g \geq 1$, we think of $(M, \omega) \in \mathcal{H}_g$ as a translation surface, that is, we consider the translation (flat) structure induced by the atlas consisting of local charts obtained by taking local primitives of ω outside its zeroes. Recall that a (maximal flat) cylinder C of (M, ω) is a maximal collection of closed parallel regular geodesics of the translation surface (M, ω) . The modulus $\text{mod}(C)$ of a cylinder C is the quotient $\text{mod}(C) = h(C)/w(C)$ of the height $h(C)$ of C (flat distance across C) by the length $w(C)$ of the waist curve of C . In this language, we say that an ergodic $SL(2, \mathbb{R})$ -invariant probability measure m on \mathcal{H}_g is *regular* if there exists a constant $K > 0$ such that

$$\lim_{\rho \rightarrow 0} \frac{m(\mathcal{H}_g(K, \rho))}{\rho^2} = 0, \quad \text{i.e.,} \quad m(\mathcal{H}_g(K, \rho)) = o(\rho^2),$$

where $\mathcal{H}_g(K, \rho)$ is the set consisting of Abelian differentials $(M, \omega) \in \mathcal{H}_g$ possessing two non-parallel cylinders C_1 and C_2 with $\text{mod}(C_i) \geq K$ and $w(C_i) \leq \rho$ for $i = 1, 2$.

Remark 1.1. A *saddle-connection* on a translation surface $(M, \omega) \in \mathcal{C}$ is a geodesic segment joining zeroes of ω without zeroes of ω in its interior. The *systole* of M , denoted by $\text{sys}(M)$, is the length of the shortest saddle-connection on M .

Each boundary component of a cylinder consists of unions of saddle-connections, so that $\text{sys}(M) \leq w(C)$ for any cylinder C of (M, ω) .

It follows from the Siegel-Veech formula (cf. [19], Theorem 2.2 in [6], and Lemma 9.1 in [5]) that we have, for small $\rho > 0$

$$m(\{\text{sys}(M) \leq \rho\}) = O(\rho^2).$$

A fortiori, those surfaces $(M, \omega) \in \mathcal{H}_g$ with a cylinder C satisfying $w(C) \leq \rho$ form a set of m -measure $O(\rho^2)$.

As it is mentioned by A. Eskin, M. Kontsevich and A. Zorich right after Definition 1 in [5], all *known* examples of ergodic $SL(2, \mathbb{R})$ -invariant probabilities m are regular: for instance, the regularity of Masur-Veech (canonical) measures was shown in Theorem 10.3 in [14] and Lemma 7.1 in [7] (see also [16] for recent related results). In particular, this led A. Eskin, M. Kontsevich and A. Zorich to conjecture (also right after Definition 1 in [5]) that *all* ergodic $SL(2, \mathbb{R})$ -invariant probabilities m on moduli spaces \mathcal{H}_g of normalized Abelian differentials are regular.

1.2. The result. In this paper, we confirm this conjecture by showing the following slightly stronger result. For $\rho > 0$, we denote by $\mathcal{H}_{g,(2)}(\rho)$ the set of $M \in \mathcal{H}_g$ which have at least two non-parallel saddle-connections of length $\leq \rho$.

Theorem 1.2. *Let m be a $SL(2, \mathbb{R})$ -invariant probability measure on \mathcal{H}_g . For small $\rho > 0$, the measure of $\mathcal{H}_{g,(2)}(\rho)$ satisfies*

$$m(\mathcal{H}_{g,(2)}(\rho)) = o(\rho^2).$$

The moduli space \mathcal{H}_g is the finite disjoint union of its strata $\mathcal{H}^{(1)}(k_1, \dots, k_\sigma)$. Here, (k_1, \dots, k_σ) runs amongst non-increasing sequences of positive integers with $\sum_{i=1}^{\sigma} k_i =$

$2g - 2$, and $\mathcal{H}^{(1)}(k_1, \dots, k_\sigma)$ consists of unit area translation surfaces $(M, \omega) \in \mathcal{H}_g$ such that the 1-form ω has σ zeroes of respective order k_1, \dots, k_σ .

Every stratum is invariant under the action of $SL(2, \mathbb{R})$. Therefore, if m is a $SL(2, \mathbb{R})$ -invariant probability measure on \mathcal{H}_g as in the theorem, we may consider its restriction to each stratum of \mathcal{H}_g (those who have positive measure) and deal separately with these restrictions.

It means that in the proof of the theorem, we may and will assume that m is supported by some stratum $\mathcal{H}^{(1)}(k_1, \dots, k_\sigma)$ of \mathcal{H}_g . Actually, we may even allow for some of the integers k_i to be equal to zero, corresponding to marked points on the translation surface which are not zeroes of the 1-form ω .

A slightly more general form of our theorem is thus:

Theorem 1.3. *Let $g > 0$, $\sigma > 0$ and let (k_1, \dots, k_σ) be a non-increasing sequence of non-negative integers with $\sum_{i=1}^{\sigma} k_i = 2g - 2$. Let $\mathcal{H}^{(1)} = \mathcal{H}^{(1)}(k_1, \dots, k_\sigma)$ be the corresponding moduli space of unit area translation surfaces. For $\rho > 0$, let $\mathcal{H}_{(2)}^{(1)}(\rho)$ be the subset consisting of translation surfaces in $\mathcal{H}^{(1)}$ which have at least two non-parallel saddle-connections of length $\leq \rho$.*

For any $SL(2, \mathbb{R})$ -invariant probability measure m on $\mathcal{H}^{(1)}$, one has

$$m(\mathcal{H}_{(2)}^{(1)}(\rho)) = o(\rho^2).$$

1.3. Reduction to a particular case. The moduli space $\mathcal{H}^{(1)}(k_1, \dots, k_\sigma)$ is only an orbifold, not a manifold, because some translation surfaces have non trivial automorphisms. We explain here how to bypass this (small) difficulty.

Denote by $\mathcal{H} = \mathcal{H}(k_1, \dots, k_\sigma)$ the moduli space of translation surfaces with combinatorial data (k_1, \dots, k_σ) of any positive area, so that $\mathcal{H}^{(1)}$ is the quotient of \mathcal{H} by the action of homotheties of positive ratio.

The moduli space $\mathcal{H}(k_1, \dots, k_\sigma)$ is the quotient of the corresponding Teichmüller space $\mathcal{Q}(k_1, \dots, k_\sigma)$ by the action of the mapping class group $MCG(g, \sigma)$. Here, a subset $\Sigma = \{A_1, \dots, A_\sigma\}$ of a (compact, connected, oriented) surface M_0 of genus g is given. The Teichmüller space $\mathcal{Q}(k_1, \dots, k_\sigma)$ is the set of structures of translation surfaces on M_0 having a zero of order k_i at A_i , up to homeomorphisms of M_0 which are isotopic to the identity rel. Σ . The mapping class group $MCG(g, \sigma)$ is the group of isotopy classes rel. Σ of homeomorphisms of M_0 preserving Σ .

The Teichmüller space $\mathcal{Q}(k_1, \dots, k_\sigma)$ is a complex manifold of dimension $2g + \sigma - 1$ with a natural affine structure given by the period map

$$\begin{aligned} \Theta : \mathcal{Q}(k_1, \dots, k_\sigma) &\longrightarrow \text{Hom}(\mathcal{H}_1(M_0, \Sigma, \mathbb{Z}), \mathbb{C}) = \mathcal{H}^1(M_0, \Sigma, \mathbb{C}) \\ \omega &\longmapsto (\gamma \longmapsto \int_{\gamma} \omega) \end{aligned}$$

which is a local homeomorphism.

The mapping class group acts properly discontinuously on Teichmüller space. However, some points in Teichmüller space may have non trivial stabilizer in $MCG(g, \sigma)$, leading to the orbifold structure for the quotient space. The stabilizer of a structure of translation surface ω on (M_0, Σ) is nothing but the finite group $\text{Aut}(M_0, \omega)$ of automorphisms of this structure. The automorphism group acts freely on the set of vertical upwards separatrices

of ω . Therefore its order is bounded by the number of such separatrices $\sum_i (k_i + 1) = 2g - 2 + \sigma$.

Let ω_0 be a structure of translation surface on (M_0, Σ) ; let G be its automorphism group, viewed as a finite subgroup of $MCG(g, \sigma)$. For ω close to ω_0 in Teichmüller space, the automorphism group $\text{Aut}(M_0, \omega)$ is contained in G , because the action of the mapping class group is properly discontinuous. Moreover, for any $g \in G$, one has $g \in \text{Aut}(M_0, \omega)$ iff $\Theta(\omega)$ is a fixed point of g (for the natural action of the mapping class group on $H^1(M_0, \Sigma, \mathbb{C})$). Thus, in a neighborhood of ω_0 , those ω which have the same automorphism group than ω_0 form an affine submanifold of Teichmüller space, and the image of this subset in the moduli space \mathcal{H} is a manifold. The intersection of this image with $\mathcal{H}^{(1)}$ is also a manifold.

For $i \geq 1$, denote by \mathcal{C}^i the set of points of $\mathcal{H}^{(1)}(k_1, \dots, k_\sigma)$ for which the automorphism group (defined up to conjugacy in the mapping class group) has order i . It follows from the previous discussion that \mathcal{C}^i is a manifold for all i , and is empty for $i > 2g - 2 + \sigma$. Moreover, each \mathcal{C}^i is invariant under the action of $SL(2, \mathbb{R})$.

Let m be a $SL(2, \mathbb{R})$ -invariant probability measure supported on $\mathcal{H}^{(1)}$. To prove the property of Theorem 1.3, it is sufficient to prove it for the restriction of m to each of the \mathcal{C}^i (those which have positive measure). Therefore we may and will assume below that m is supported on some \mathcal{C}^i .

1.4. Scheme of the proof. The basic idea of the proof of Theorem 1.3 is the following. We consider, in a level set $\{\text{sys}(M) = \rho_0\}$ of the systole function on \mathcal{C}^i , the subset of translation surfaces whose shortest saddle-connections are all parallel. We will use the $SL(2, \mathbb{R})$ -action to move relatively large parts of this subset to further deep sublevels $\{\text{sys}(M) \leq \rho := \rho_0 \exp(-T)\}$ for $T \gg 1$. The translation surfaces obtained in this way have the property that all saddle-connections not parallel to a minimizing one are much longer than the systole. A computation in appropriate pieces of $SL(2, \mathbb{R})$ -orbits, on which the invariant probability desintegrates as Haar measure, shows that most surfaces in the sublevel $\{\text{sys}(M) \leq \rho\}$ are obtained in this way. Thus we can conclude that the complement in \mathcal{C}^i , which contains the intersection $\mathcal{H}_{g,(2)}(\rho) \cap \mathcal{C}^i$ has measure $o(\rho^2)$.

The article is organized as follows. In Section 2, we review some material on Rokhlin's desintegration theorem ([17], see also [21]), which allows to consider separately each orbit of the action of $SL(2, \mathbb{R})$. The statement that we aim at (Proposition 2.6) is well-known to specialists, but we were not able to find a proper reference in the literature.

In Section 3, we discuss a couple of elementary facts about $SL(2, \mathbb{R})$ and its action on \mathbb{R}^2 (see Proposition 3.1, Proposition 3.3 and Proposition 3.4) lying at the heart of our "orbit by orbit" estimates.

In Section 4, we construct from the invariant probability m a related measure m_0 which has finite total mass and is supported on the subset X_0^* of the level set $\{\text{sys}(M) = \rho_0\}$ formed by surfaces on which all minimizing saddle-connections are vertical. This measure m_0 enters in the formula (Corollary 4.2) for the m -measure of certain subsets of the sublevels $\{\text{sys}(M) \leq \rho_0 \exp(-T)\}$.

In Section 5, we show that the measure of slices of the form $\{\rho_0 \geq \text{sys}(M) \geq \rho_0 \exp(-\tau)\}$ when τ is small is related to the total mass of m_0 (Corollary 5.4). The slice is divided into a *regular part*, obtained by pushing X_0^* into the slice through the action of $SL(2, \mathbb{R})$, and a *singular part*, whose measure is much smaller (Proposition 5.3).

In Section 6, we bring together the results of the previous sections to present the proof of the theorem.

1.5. Notations. We will assume some familiarity with the basic features of Abelian differentials, translation surfaces and their moduli spaces: we refer the reader to the surveys [24] and [20] for gentle introductions to the subject. We will use the following notations:

- \mathcal{C} denotes what was denoted by \mathcal{C}^i above, namely the subset of a moduli space $\mathcal{H}^{(1)}(k_1, \dots, k_\sigma)$ formed by points whose stabilizer in the mapping class group has a given order.
- m is a $SL(2, \mathbb{R})$ -invariant probability measure supported on the manifold \mathcal{C} .
- (e_1, e_2) denotes the standard basis of \mathbb{R}^2 and $\|\cdot\|$ is the usual Euclidean norm on \mathbb{R}^2 .
- $R_\theta \in SO(2, \mathbb{R})$ is the rotation of angle θ .
- g_t is the diagonal matrix $\text{diag}(e^t, e^{-t})$ in $SL(2, \mathbb{R})$.
- n_u is the lower triangular matrix $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$.
- $N_{a,b}$ is the upper triangular matrix $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$.

1.6. Two basic facts. We use several times the following lemma, which is an immediate consequence of Fubini's theorem.

Lemma 1.4. *Let (X, \mathcal{B}, m) be a probability space and let G be a locally compact group acting measurably on X by measure-preserving automorphisms, and let ν be some Borel probability measure on G . For any measurable subset $B \subset X$, one has*

$$m(B) = \int_X \nu(\{g \in G, g.x \in B\}) dm(x).$$

We will apply this taking for ν the normalized restriction of a Haar measure on G to some compact subset.

We will use in the proof of Lemma 5.6 the following fundamental fact on moduli space of translation surfaces.

For $\rho > 0$, $R > 0$, and any $(M, \omega) \in \mathcal{H}^{(1)}(k_1, \dots, k_\sigma)$ with $\text{sys}(M) \geq \rho$, the number of holonomy vectors of saddle-connections of (M, ω) of norm $\leq R$ is bounded by a constant which depends only on ρ , R and k_1, \dots, k_σ . See [6], Theorem 5.4 (and also [13]), for a much stronger result.

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2. CONDITIONAL MEASURES

2.1. The general setting. We start by recalling the content of Rokhlin's disintegration theorem ([17], see also [21]).

Definition 2.1. A probability space (X, \mathcal{B}, m) is *Lebesgue* if either it is purely atomic or its continuous part is isomorphic mod.0 to $[0, a]$ equipped with the standard Lebesgue measure. Here $a \in (0, 1]$ is the total mass of the continuous part of m .

Definition 2.2. A *Polish space* is a topological space homeomorphic to a separable complete metric space.

Any open or closed subset of a Polish space is Polish.

Any Borel probability measure on a Polish space is Lebesgue. We generally omit in this case the reference to the σ -algebra \mathcal{B} , which is the σ -algebra generated by the Borel sets and the subsets of Borel sets of measure 0.

For a partition ζ of a set X , we denote by $\zeta(x)$ the element of ζ that contains a point $x \in X$.

Definition 2.3. Let (X, \mathcal{B}, m) be a Lebesgue probability space. A *measurable partition* of X is a partition ζ of X which is the limit of a monotonous sequence $(\zeta_n)_{n \geq 0}$ of finite partitions by elements of \mathcal{B} . This means that, for all $x \in X$, $n \geq 0$, one has

$$\zeta_n(x) \in \mathcal{B}, \quad \zeta_{n+1}(x) \subset \zeta_n(x), \quad \zeta(x) = \bigcap_{n \geq 0} \zeta_n(x).$$

As (X, \mathcal{B}, m) is Lebesgue, the partition of X by points is measurable (mod.0).

Definition 2.4. Let ζ be a measurable partition of X . A *system of conditional measures* for $(X, \mathcal{B}, m, \zeta)$ is a family $(m_x)_{x \in X}$ of probability measures on (X, \mathcal{B}) satisfying the following properties:

- For any $x \in X$, one has $m_x(\zeta(x)) = 1$.
- For any $x, y \in X$ such that $\zeta(x) = \zeta(y)$, one has $m_x = m_y$.
- For any $B \in \mathcal{B}$, the function $x \mapsto m_x(B)$ is measurable and one has

$$m(B) = \int_X m_x(B) dm(x).$$

The content of Rokhlin's theorem is that such a system of conditional measures always exists, and is essentially unique in the following sense: if $(m'_x)_{x \in X}$ is another such system, then $m_x = m'_x$ for m -a.a x .

2.2. A special setting. Let X be a Polish space, and let G be a Lie group. We will denote by ν some given left invariant Haar measure on G , by d some given left invariant Riemannian distance on G , and by p_1 the canonical projection from $X \times G$ onto X . We let G act on the left on $X \times G$ by $g.(x, h) := (x, gh)$, i.e the product of the trivial action by the standard action.

Let Z be a non empty open subset of $X \times G$, and let m be a Borel probability measure on Z . As an open subset of a Polish space, Z is a Polish space. Thus (Z, m) is a Lebesgue probability space.

We will denote by m_1 the Borel probability measure $(p_1)_*m$ on the open subset $p_1(Z)$ of X , and by $Z_x = \{x\} \times U_x$ the fiber of Z over x . Here U_x is an open subset of G . The partition ζ of Z defined by $\zeta(x, g) = Z_x$ is measurable.

Let $(m_{(x,g)})_{(x,g) \in Z}$ be a system of conditional measures for (Z, m, ζ) . From the second property in the definition of conditional measures, the measure $m_{(x,g)}$ does not depend on the second variable g . We will write m_x instead of $m_{(x,g)}$. For each $x \in p_1(Z)$, m_x may be seen as a probability measure on U_x .

Definition 2.5. The Borel probability measure m on Z is *invariant* if, for any measurable subset $W \subset Z$, any $g \in G$ such that $g.W \subset Z$, one has $m(g.W) = m(W)$.

Proposition 2.6. *Assume that m is invariant. Then, for m_1 -a.e x , the non empty open set U_x has finite Haar measure and we have*

$$m_x = \frac{1}{\nu(U_x)} \nu|_{U_x}.$$

Proof. Choose a countable dense subset $D \subset G$ and denote by \mathbb{B} the set of closed balls (for the distance d) in G with center at a point of D and positive rational radius. The proof of the proposition is an easy consequence of the following elementary lemma, whose proof will be given afterwards.

Lemma 2.7. *Let U be a non empty open subset of G and let μ be a probability measure on U . Assume that, for any $B \in \mathbb{B}$, $g \in D$ such that both B and $g.B$ are contained in U , one has $\mu(g.B) = \mu(B)$. Then U has finite Haar measure and $\mu = \frac{1}{\nu(U)} \nu|_U$.*

In view of the conclusion of the lemma, the conclusion of the proposition will follow if we know that, for m_1 -a.e x , the measure m_x has the invariance property stated in the hypothesis of the lemma. Let $B \in \mathbb{B}$, $g \in D$. The set $X(B, g)$ formed of $x \in X$ such that both B and $g.B$ are contained in U_x is open in X . For any m_1 -measurable $Y \subset X(B, g)$, one has, as m is invariant,

$$\begin{aligned} \int_Y m_x(B) dm_1(x) &= m(Y \times B) \\ &= m(Y \times g.B) \\ &= \int_Y m_x(g.B) dm_1(x). \end{aligned}$$

It follows that $m_x(B) = m_x(g.B)$ for m_1 -a.a $x \in X(B, g)$. Taking a countable intersection over the possible B, g gives the assumption of the lemma so the proof of the proposition is complete. \square

Proof of lemma 2.7. We first show that μ is absolutely continuous w.r.t. the Haar measure ν , i.e belongs to the Lebesgue class. Let B be any closed ball in \mathbb{B} of small radius $r > 0$ contained in U . There exists an integer $M \geq Cr^{-\dim G}$ (where the constant $C > 0$ depends on U but not on B) and elements $g_1, \dots, g_M \in D$ such that the balls $g_i.B$ are disjoint and contained in U . Then we have, from the assumption of the lemma

$$M\mu(B) = \mu\left(\bigcup_{i=1}^M g_i.B\right) \leq \mu(U) = 1.$$

It follows that $\mu(B) \leq C^{-1}r^{\dim G}$ for all $B \in \mathbb{B}$ contained in U . This implies that μ belongs to the Lebesgue class and that its density ϕ w.r.t. the Haar measure ν is bounded. By the Lebesgue density theorem, for ν -almost all $x \in U$, one has

$$\phi(x) = \lim_{B \ni x} \frac{\mu(B)}{\nu(B)}$$

where the limit is taken over balls in \mathbb{B} containing x with radii converging to 0. If x, x' are any two points of U with this property, one can find a sequence B_i of balls in \mathbb{B} with radii converging to 0 and a sequence $g_i \in D$ such that $x \in B_i \subset U$ and $x' \in g_i.B_i \subset U$ for all i . It follows from the invariance assumption on μ and the left invariance of Haar measure that $\phi(x) = \phi(x')$. Thus the density ϕ is constant ν -almost everywhere in U and the proof of the lemma is complete. \square

Remark 2.8. In the defining property of conditional measures

$$\int_Z f(x, g) dm(x, g) = \int_{p_1(Z)} \left(\int_{U_x} f(x, g) dm_x(g) \right) dm_1(x)$$

both m_1 and m_x are **probability** measures. In the context of Proposition 2.6, one may also write

$$\int_Z f(x, g) dm(x, g) = \int_{p_1(Z)} \left(\int_{U_x} f(x, g) d\nu(g) \right) (\nu(U_x))^{-1} dm_1(x).$$

However, the measure $(\nu(U_x))^{-1} dm_1(x)$ is not necessarily finite.

3. PRELIMINARIES ON $SL(2, \mathbb{R})$

3.1. The decomposition $g_t R_\theta n_u$. Let W be the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL(2, \mathbb{R})$ such that $d > 0$ and $|bd| < \frac{1}{2}$.

Proposition 3.1. *The map $(t, \theta, u) \mapsto g_t R_\theta n_u$ from $\mathbb{R} \times (-\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{R}$ is a diffeomorphism onto W . Using this map to take (t, θ, u) as coordinates on W , the restriction to W of (a conveniently scaled version of) the Haar measure is equal to $\cos 2\theta dt d\theta du$.*

Proof. The vertical basis vector e_2 is fixed by n_u . For $|\theta| < \frac{\pi}{4}$ and $t \in \mathbb{R}$, its image (b, d) under $g_t R_\theta$ satisfy $d > 0$ and $|bd| < \frac{1}{2}$.

Conversely, given (b, d) satisfying these conditions, there is a unique pair $(t, \theta) \in \mathbb{R} \times (-\frac{\pi}{4}, \frac{\pi}{4})$, depending smoothly on (b, d) , such that $(b, d) = g_t R_\theta \cdot e_2$. The first assertion of the proposition follows.

Write the restriction to W of the Haar measure on $SL(2, \mathbb{R})$ as $\psi(t, \theta, u) dt d\theta du$, for some positive smooth function ψ on $\mathbb{R} \times (-\frac{\pi}{4}, \frac{\pi}{4}) \times \mathbb{R}$. As the Haar measure is left-invariant and right-invariant, the function ψ does not depend on t and u , only on θ . A small calculation, using the left-invariance under R_θ , shows that $\psi(\theta) = \cos 2\theta$. \square

3.2. Euclidean norms along a $SL(2, \mathbb{R})$ -orbit. The following lemma is elementary and well-known.

Lemma 3.2. *The map $(\theta, a, b) \mapsto R_\theta N_{a,b}$ from $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}_{>0} \times \mathbb{R}$ to $SL(2, \mathbb{R})$ is a diffeomorphism. The measure $d\theta da db$ is sent to a Haar measure by this diffeomorphism.*

Proposition 3.3. *Let v, v' be vectors in \mathbb{R}^2 with $v \neq \pm v'$. The Haar measure of the set $E(v, v', \tau)$ consisting of the elements $\gamma \in SL(2, \mathbb{R})$ such that*

$$\|\gamma\| \leq 2, \quad \exp(-\tau) \leq \|\gamma \cdot v\| \leq \exp \tau, \quad \exp(-\tau) \leq \|\gamma \cdot v'\| \leq \exp \tau$$

is $O(\tau^{\frac{3}{2}})$ when τ is small; the implied constants are uniform when $\|v \pm v'\|$ is bounded away from 0.

Proof. The set $E(v, v', \tau)$ is empty (for small τ) unless $\frac{1}{3} \leq \|v\| \leq 3$, $\frac{1}{3} \leq \|v'\| \leq 3$, so we may assume that v, v' are constrained by these inequalities. We may also assume that $v = (p, 0)$ with $\frac{1}{3} \leq p \leq 3$. We write $v' = (q, r)$, and $\gamma = R_\theta N_{a,b}$ as in the Lemma. The function

$$N : \gamma \mapsto (\|\gamma \cdot v\|^2, \|\gamma \cdot v'\|^2)$$

does not depend on θ ; one has

$$N(a, b) = (p^2 a^2, (qa + rb)^2 + r^2 a^{-2}).$$

The Jacobian matrix of N is

$$2 \begin{pmatrix} p^2 a & 0 \\ q(qa + rb) - r^2 a^{-3} & r(qa + rb) \end{pmatrix}.$$

As $a > 0$, this matrix is invertible unless $r(qa + rb) = 0$, i.e. $\gamma.v, \gamma.v'$ are collinear or orthogonal. We conclude as follows

- Assume first that $v' = \pm(\lambda v + w)$ with $0 < \lambda \neq 1$, $\|w\| < \frac{1}{100}|\lambda - 1|$. Then $\gamma.v' = \pm(\lambda\gamma.v + \gamma.w)$ with $\|\gamma.w\| < \frac{2}{100}|\lambda - 1|$ when $\|\gamma\| \leq 2$. As $\lambda \neq 1$, we cannot have at the same time

$$\exp(-\tau) \leq \|\gamma.v\| \leq \exp \tau, \quad \exp(-\tau) \leq \|\gamma.v'\| \leq \exp \tau$$

if τ is small enough. The set $E(v, v', \tau)$ is thus empty for small τ ; the implied constant depends only on $|\lambda - 1|$, i.e. on $\|v \pm v'\|$.

- Assume next that v, v' are orthogonal, i.e. $q = 0$. Then $N(a, b) = (p^2 a^2, r^2(b^2 + a^{-2}))$. The condition $\exp(-\tau) \leq \|\gamma.v\| \leq \exp \tau$ determines an interval for a of length $O(\tau)$. For each value of a in this interval, the condition $\exp(-\tau) \leq \|\gamma.v'\| \leq \exp \tau$ determines an interval for b whose length is $O(\tau^{\frac{1}{2}})$ (and exactly of order $\tau^{\frac{1}{2}}$ in the worst case $pr = 1$). Thus, the Haar measure of $E(v, v', \tau)$ is $O(\tau^{\frac{3}{2}})$. Observe that this is still true if we relax the condition $\|\gamma\| \leq 2$ to $\|\gamma\| \leq 6$.
- Assume that there exists $\gamma_0 \in SL(2, \mathbb{R})$ such that $\|\gamma_0\| \leq 3$ and $\gamma_0.v, \gamma_0.v'$ are orthogonal. Then the required estimate is a consequence of the previous case after translating by γ_0 , taking into account the observation concluding the previous discussion.
- Finally, assume that none of the above holds. Then the Jacobian matrix of N on $\|\gamma\| \leq 2$ is everywhere invertible and the norm of its inverse is uniformly bounded when $\|v \pm v'\|$ is bounded away from 0. In this case, the Haar measure of $E(v, v', \tau)$ is $O(\tau^2)$, with a uniform implied constant when $\|v \pm v'\|$ is bounded away from 0.

The proposition is proved. \square

3.3. On the action of the diagonal subgroup. Let R_θ be some given rotation. For $T > 0$, consider the set

$$J(T, \theta) := \{t \in \mathbb{R}, \|g_t R_\theta.e_2\| < \exp(-T)\}.$$

Proposition 3.4. *The set $J(T, \theta)$ is empty iff $|\sin 2\theta| \geq \exp(-2T)$. When $|\sin 2\theta| < \exp(-2T)$, writing $\sin 2\theta = e^{-2T} \sin \omega$ with $\cos \omega > 0$, the set $J(T, \theta)$ is an open interval of length $\frac{1}{2} \log \frac{1+\cos \omega}{1-\cos \omega}$.*

Proof. A real number t belongs to $J(T, \theta)$ iff

$$e^{2t} \sin^2 \theta + e^{-2t} \cos^2 \theta < e^{-2T}.$$

Thus $J(T, \theta)$ is empty unless

$$\Delta := e^{-4T} - \sin^2 2\theta > 0,$$

which gives the first assertion of the proposition.

When this condition holds, we write $\sin 2\theta = e^{-2T} \sin \omega$ with $\cos \omega > 0$. One has $\Delta^{\frac{1}{2}} = e^{-2T} \cos \omega$, which implies through an elementary calculation the second assertion of the proposition. \square

For later use, we note that

Lemma 3.5.

$$\int_0^{\frac{\pi}{2}} \log \frac{1 + \cos \omega}{1 - \cos \omega} \cos \omega \, d\omega = \pi.$$

Proof. The change of variables $u = \tan \frac{\omega}{2}$ transforms the given integral into $4 \int_0^1 \log u^{-1} \frac{1-u^2}{(1+u^2)^2} du$.

We then have, as $\int_0^1 u^n \log u^{-1} du = (n+1)^{-2}$ for $n \geq 0$

$$\begin{aligned} \int_0^1 \log u^{-1} \frac{1-u^2}{(1+u^2)^2} du &= \int_0^1 \log u^{-1} \sum_{n \geq 0} (-1)^n (2n+1) u^{2n} du \\ &= \sum_{n \geq 0} \frac{(-1)^n}{2n+1} \\ &= \frac{\pi}{4} \end{aligned}$$

□

4. CONSTRUCTION OF A MEASURE RELATED TO m

In the next three sections, the setting is as indicated in subsection 1.3 and subsection 1.5: \mathcal{C} is a $SL(2, \mathbb{R})$ -invariant **manifold** contained in moduli space, and m is a $SL(2, \mathbb{R})$ -invariant probability measure supported on \mathcal{C} .

Until further notice, we fix some number $\rho > 0$, small enough so that the m -measure of $\{M \in \mathcal{C} : \text{sys}(M) > \rho\}$ is positive.

Let X be the level set $\{M \in \mathcal{C} : \text{sys}(M) = \rho\}$. Let X_0^* be the subset of X formed of surfaces M for which all non-vertical saddle-connections have length $> \rho$. Let $X^* := \bigcup_{\theta} R_{\theta}(X_0^*)$. Observe that in this union, one has $R_{\theta}(X_0^*) = R_{\theta+\pi}(X_0^*)$ but also $R_{\theta_0}(X_0^*) \cap R_{\theta_1}(X_0^*) = \emptyset$ for $-\frac{\pi}{2} < \theta_0 < \theta_1 \leq \frac{\pi}{2}$. Thus we have

$$X^* = \bigsqcup_{-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}} R_{\theta}(X_0^*).$$

The subsets X^* and X_0^* are submanifolds of \mathcal{C} of codimension one and two respectively.

Observe that for $|\theta| < \frac{\pi}{4}$, $g_t R_{\theta} e_2$ is shorter than e_2 for $0 < t < \log \cot |\theta|$. It follows that $g_t R_{\theta}(X_0^*)$ is disjoint from X for $0 < t < \log \cot |\theta|$, and thus the union

$$Y^* := \bigcup_{|\theta| < \frac{\pi}{4}} \bigcup_{0 < t < \log \cot |\theta|} g_t R_{\theta}(X_0^*) = \bigsqcup_{|\theta| < \frac{\pi}{4}} \bigsqcup_{0 < t < \log \cot |\theta|} g_t R_{\theta}(X_0^*)$$

is a **disjoint** union.

In the following proposition, we use this to identify Y^* with $\{(t, \theta, M) \in \mathbb{R} \times (-\frac{\pi}{4}, \frac{\pi}{4}) \times X_0^*, 0 < t < \log \cot |\theta|\}$.

Proposition 4.1. *The m -measure of Y^* is positive. Moreover, there exists a finite measure m_0 on X_0^* such that the restriction of m to Y^* satisfies*

$$m|_{Y^*} = dt \times \cos 2\theta \, d\theta \times m_0.$$

Proof. As n_u fixes the vertical basis vector e_2 , the vectorfield which is the infinitesimal generator of the action on \mathcal{C} of the one-parameter group n_u is tangent to X_0^* at any point of X_0^* .

Let $M \in \mathcal{C}$ be such that $\text{sys}(M) > \rho$. We claim the set of $\gamma \in SL(2, \mathbb{R})$ such that $\gamma.M \in Y^*$ has non empty interior, hence positive Haar measure. It then follows from Lemma 1.4 and the hypothesis on ρ that $m(Y^*) > 0$.

To prove the claim, it is sufficient to check that there exists $\gamma_0 \in SL(2, \mathbb{R})$ such that $\gamma_0.M \in X_0^*$, because then $g_t R_\theta n_u \gamma_0.M \in Y^*$ for small u , $|\theta| < \frac{\pi}{4}$, and $0 < t < \log \cot |\theta|$. This element γ_0 can be taken as $\gamma_0 = g_s R_\omega$, where ω is such that the systole for $R_\omega.M$ is realized in the vertical direction, and $s = \log \frac{\text{sys}(M)}{\rho}$. This proves the first assertion of the proposition.

Let Σ be a smooth codimension-one submanifold of X_0^* which is transverse to this vectorfield. Taking Σ small enough, there exists $u_0 > 0$ such that $n_u.\Sigma \subset X_0^*$ for $|u| < u_0$ and $n_u.\Sigma \cap \Sigma = \emptyset$ for $0 < u < 2u_0$.

Then the map

$$\Psi_0(u, M) := n_u.M$$

is an smooth diffeomorphism from $(-u_0, u_0) \times \Sigma$ onto a subset U of X_0^* and the map

$$\Psi(t, \theta, u, M) := g_t R_\theta n_u.M$$

is an smooth diffeomorphism from

$$\{(t, \theta, u, M) \in \mathbb{R} \times (-\frac{\pi}{4}, \frac{\pi}{4}) \times (-u_0, u_0) \times \Sigma, 0 < t < \log \cot |\theta|\}$$

onto an open subset V of \mathcal{C} . Moreover, one can choose a locally finite covering of X_0^* by such sets U . The measure m_0 will be defined by its restriction to such sets.

If the m -measure of V is 0, the set U will be disjoint from the support of m_0 .

Assume that $m(V) > 0$. As m is $SL(2, \mathbb{R})$ -invariant, the measure $\Psi^*(m|_V)$ can be written, using Proposition 2.6 (with $G = SL(2, \mathbb{R})$) and Proposition 3.1, as

$$dt \times \cos 2\theta \, d\theta \times du \times \nu$$

for some finite measure ν on Σ (cf. Remark 2.8: the open subset U_x of Proposition 2.6 is here independent of x).

One thus defines

$$m_{0|U} = (\Psi_0)_*(du \times \nu).$$

Then m_0 satisfies the required conditions. \square

In the next corollary, $J(T, \theta)$ is the interval that has been defined in Proposition 3.4.

Corollary 4.2. *Let B be a Borel subset of X_0^* , $\omega_0 \in (0, \frac{\pi}{2}]$, $T > 0$. Let $Y(T, \omega_0, B)$ be the set of surfaces $M' = g_t R_\theta M$ in Y^* such that $M \in B$, $|\sin 2\theta| < \exp(-2T) \sin \omega_0$ and $t \in J(T, \theta)$. One has*

$$m(Y(T, \omega_0, B)) = \frac{1}{4} \exp(-2T) m_0(B) \int_{-\omega_0}^{\omega_0} \log \frac{1 + \cos \omega}{1 - \cos \omega} \cos \omega \, d\omega$$

Proof. Write $\sin 2\theta = e^{-2T} \sin \omega$ as in Proposition 3.4, so that the condition $|\sin 2\theta| < \exp(-2T) \sin \omega_0$ is equivalent to $|\omega| < \omega_0$. As the length of $J(T, \theta)$ is

$$\frac{1}{2} \log \frac{1 + \cos \omega}{1 - \cos \omega}$$

we obtain from Proposition 4.1

$$\begin{aligned} m(Y(T, \omega_0, B)) &= m_0(B) \int_{|\sin 2\theta| < \exp(-2T) \sin \omega_0} |J(T, \theta)| \cos 2\theta d\theta \\ &= \frac{1}{4} \exp(-2T) m_0(B) \int_{-\omega_0}^{\omega_0} \log \frac{1 + \cos \omega}{1 - \cos \omega} \cos \omega d\omega. \end{aligned}$$

□

5. MEASURE OF THE SLICE $\{\rho \geq \text{sys}(M) \geq \rho \exp(-\tau)\}$

For $\rho > 0$, we denote by $F(\rho)$ the m -measure of the set $\{\text{sys}(M) \leq \rho\}$. This is a non-decreasing function of ρ .

For any $\rho > 0$, the level set $\{\text{sys}(M) = \rho\}$ has m -measure 0, as may be seen for instance by an elementary application of Lemma 1.4. It follows that the function F is continuous. We will prove in this section (see Corollary 5.4) that, for any ρ such that $F(\rho) < 1$, the function F has at ρ a positive left-derivative $F'(\rho)$ which is equal to $\pi \rho^{-1} m_0(X_0^*)$, where X_0^* , m_0 are as Proposition 4.1.

In this section, ρ is as above a positive number such that $F(\rho) < 1$.

5.1. The regular part of the slice. For $M \in X^*$ and $t \geq 0$, we define

$$\Phi_t(M) := R_\theta g_t R_{-\theta}(M), \quad \text{when } M \in R_\theta(X_0^*).$$

Observe that the two possible choices for θ give the same result as R_π is the non trivial element of the center of $SL(2, \mathbb{R})$.

The systole of $\Phi_t(M)$ is equal to $\rho \exp(-t)$. For $M \in R_\theta(X_0^*)$, all saddle connections of $\Phi_t(M)$ with minimal length have the same angle θ with the vertical direction. It follows that the Φ_t are injective, and that $\Phi_t(X^*) \cap \Phi_{t'}(X^*) = \emptyset$ for $t \neq t'$.

Definition 5.1. For $\tau > 0$, the set $\bigsqcup_{0 \leq t \leq \tau} \Phi_t(X^*)$ is called the *regular part* of the slice

$$S(\tau) := \{\rho \geq \text{sys}(M) \geq \rho \exp(-\tau)\}.$$

Its complement (in $S(\tau)$) is called the *singular part* of the slice $S(\tau)$.

Next, we define, for a Borel subset B of X^* and $\tau > 0$

$$\tilde{m}_\tau(B) := \frac{2}{1 - \exp(-2\tau)} m\left(\bigsqcup_{0 \leq t \leq \tau} \Phi_t(B)\right).$$

Proposition 5.2. *The measure \tilde{m}_τ is independent of τ , and is equal to the product $d\theta \times m_0$.*

Proof. As the measure \tilde{m}_τ on X^* is invariant under the action on X^* of the group of rotations, it follows from Proposition 2.6 that the measure \tilde{m}_τ can be written as $d\theta \times m_\tau$, for some finite measure m_τ on X_0^* . We will show that $m_\tau = m_0$.

Let Σ and u_0 be as in the proof of Proposition 4.1. For $(u, M) \in (-u_0, u_0) \times \Sigma$, any θ and any $t \geq 0$, we have

$$\Phi_t(R_\theta n_u.M) = R_\theta g_t n_u.M.$$

Observe that for $t \geq 0$, $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$, the element $R_\theta g_t$ belongs to the set W of subsection 3.1, so that we can write, according to Proposition 3.1

$$R_\theta g_t = g_{T(t,\theta)} R_{\Theta(t,\theta)} n_{U(t,\theta)}$$

for some smooth functions T, Θ, U . For θ close to 0, we have

$$T(t, \theta) = t + O(\theta), \quad U(t, \theta) = O(\theta), \quad \Theta(t, \theta) = e^{-2t}\theta + O(\theta^2).$$

Let $B_0 = (u_1, u_2) \times B$ be an elementary Borel subset of $(-u_0, u_0) \times \Sigma \subset X_0^*$, where B is a Borel subset of Σ . For small $\theta > 0$, we have, in view of the formula for Haar measure in Proposition 3.1

$$\begin{aligned} \tilde{m}_\tau([0, \theta] \times B_0) &= \frac{2}{1 - \exp(-2\tau)} \int_B \int_{u_1}^{u_2} \int_0^\tau e^{-2t} \theta \, dt \, du \, d\nu + O(\theta^2) \\ &= m_0(B_0) \theta + O(\theta^2). \end{aligned}$$

This proves that $m_\tau = m_0$. \square

5.2. Estimate for the singular part of the slice. Let $Z(\tau)$ be the subset of $S(\tau)$ consisting of surfaces M having a saddle-connection of length $\leq \rho \exp \tau$ which is not parallel to a minimizing one. Clearly the singular part of the slice $S(\tau)$ is contained in $Z(\tau)$.

Proposition 5.3. *For small $\tau > 0$, the m -measure of $Z(\tau)$ (hence also the m -measure of the singular part of the slice $S(\tau)$) is $o(\tau)$.*

From Propositions 5.2 and 5.3, we get the

Corollary 5.4. *One has*

$$\lim_{\tau \rightarrow 0} \frac{1}{\tau} m(\{\rho \geq \text{sys}(M) \geq \rho \exp(-\tau)\}) = \pi m_0(X_0^*).$$

Proof of Proposition 5.3. For $M \in S(\tau)$, let us denote by $\hat{\theta}(M)$ the smallest non-zero angle between two saddle-connections with length $\leq 3\rho$ (If all connections with length $\leq 3\rho$ are parallel, we define $\hat{\theta}(M) = \frac{\pi}{2}$). The required estimate for the measure of $Z(\tau)$ follows from two lemmas:

Lemma 5.5. *For any $\eta > 0$, there exists $\hat{\theta}_0$ such that, for any $\tau > 0$ small enough, one has*

$$m(\{M \in S(\tau) : \hat{\theta}(M) < \hat{\theta}_0\}) < \eta\tau.$$

Lemma 5.6. *For any $\hat{\theta}_0$, one has*

$$m(\{M \in Z(\tau) : \hat{\theta}(M) \geq \hat{\theta}_0\}) = O(\tau^{\frac{3}{2}}),$$

where the implied constant depends on $\hat{\theta}_0, \rho$ and g .

Proof of Lemma 5.5. Denote by $S_1(\tau)$ the subset of $S(\tau)$ consisting of translation surfaces for which there exists a length-minimizing saddle-connection whose direction form an angle $\leq \frac{\pi}{6}$ with the vertical direction.

It follows from Lemma 1.4 (with $G = SO(2, \mathbb{R})$ equipped with the Haar measure) that for any $SO(2, \mathbb{R})$ -invariant subset $\mathcal{S} \subset S(\tau)$, we have

$$m(\mathcal{S}) \leq 3 m(S_1(\tau) \cap \mathcal{S}).$$

We will apply this relation with

$$\mathcal{S} = \{M \in S(\tau), \hat{\theta}(M) < \hat{\theta}_0\},$$

for some appropriate $\hat{\theta}_0$.

For $M \in S_1(\tau)$, and any positive integer j such that

$$\exp(3j\tau) < \cot \frac{\pi}{6} = \sqrt{3}$$

the systole of $g_{3j\tau}.M$ is $< \rho \exp(-\tau)$. Therefore the images of $S_1(\tau)$ under the elements $g_{3j\tau}$, for $0 \leq j < \frac{\log 3}{6\tau} - 1$, are disjoint. Observe also that for such M, j , the surface $M' := g_{3j\tau}.M$ has a systole in $(\frac{\rho}{2}, \sqrt{3}\rho)$ and has two saddle-connections of length at most $3\sqrt{3}\rho$ with a non-zero angle $\leq A\hat{\theta}(M)$, for some absolute constant $A > 1$.

Let $\eta > 0$. Let $\hat{\theta}_1 > 0$ be small enough in order that the m -measure of the set of surfaces M' with systole in $(\frac{\rho}{2}, \sqrt{3}\rho)$, having two non-parallel saddle-connexions of length $\leq 3\sqrt{3}\rho$ and angle $< \hat{\theta}_1$, is $< \frac{\eta}{18}$. Choosing $\hat{\theta}_0 := A^{-1}\hat{\theta}_1$, as the number of values of j with $0 \leq j < \frac{\log 3}{6\tau} - 1$ is $\geq \frac{1}{6\tau}$ for τ small enough, we obtain

$$m(\{M \in S(\tau), \hat{\theta}(M) < \hat{\theta}_0\} \cap S_1) < 6\tau \cdot \frac{\eta}{18},$$

$$m(\{M \in S(\tau), \hat{\theta}(M) < \hat{\theta}_0\}) < \eta\tau.$$

□

Proof of Lemma 5.6. We will apply Lemma 1.4 with

$$B := \{M \in Z(\tau), \hat{\theta}(M) \geq \hat{\theta}_0\},$$

$G = SL(2, \mathbb{R})$, and ν the normalized restriction of a Haar measure to

$$K := \{\gamma \in SL(2, \mathbb{R}), \|\gamma\| \leq 2\},$$

where $\|\cdot\|$ is the Euclidean operator norm. We therefore have to estimate the relative measure in K of the sets $\{\gamma \in K, \gamma.M \in B\}$.

If $K.M$ does not intersect B , this measure is 0.

Let $M \in \mathcal{C}$, $\gamma \in K$ such that $\gamma.M \in B$. As $\|\gamma\| = \|\gamma^{-1}\| \leq 2$, we have

$$\frac{1}{2}\rho \exp(-\tau) \leq \text{sys}(M) \leq 2\rho.$$

As $B \subset Z(\tau)$, there exist non colinear holonomy vectors v, v' of saddle connections of M such that

$$\rho \exp(-\tau) \leq \|\gamma.v\| \leq \rho \exp(\tau), \quad \rho \exp(-\tau) \leq \|\gamma.v'\| \leq \rho \exp(\tau).$$

This means that γ belongs to the set $E(\rho^{-1}v, \rho^{-1}v', \tau)$ of Proposition 3.3.

As $\|\gamma\| = \|\gamma^{-1}\| \leq 2$, we must have (for small τ) $\|v\| \leq 3\rho, \|v'\| \leq 3\rho$.

Moreover, the angle between the directions of v , v' is $\geq A^{-1}\hat{\theta}_0$ for some appropriate absolute constant $A > 1$: otherwise, we would have $\hat{\theta}(\gamma.M) < \hat{\theta}_0$, in contradiction to $\gamma.M \in B$. This imply that $\|\rho^{-1}(v \pm v')\|$ is bounded from below by a constant c depending only on $\hat{\theta}_0$.

Let v_1, \dots, v_N be the holonomy vectors of saddle-connections of M of lengths $\leq 3\rho$.

We have shown that

$$\{\gamma \in K : \gamma.M \in B\} \subset \bigcup E(\rho^{-1}v_i, \rho^{-1}v_j, \tau)$$

where the union is taken over indices i, j such that $\|\rho^{-1}(v_i \pm v_j)\| \geq c$. In view of Proposition 3.3, we obtain

$$\frac{\nu(\{\gamma \in K : \gamma.M \in B\})}{\nu(K)} \leq N^2 \cdot O_{\hat{\theta}_0}(\tau^{3/2})$$

By the fundamental fact recalled in Subsection 1.6, the integer N has an upper bound depending only on ρ and g .

The statement in Lemma 5.6 follows from this estimate, plugged into Lemma 1.4. \square

This completes the proof of Proposition 5.3. \square

6. PROOF OF THEOREM 1.3

We recall that $F(\rho)$ stands for the m -measure of the set $\{sys(M) \leq \rho\}$. It is a continuous non-decreasing function of $\rho > 0$. At any $\rho > 0$ such that $F(\rho) < 1$, the function F has a **positive** left-derivative $F'(\rho)$ which is equal to $\pi\rho^{-1}m_0(X_0^*)$ (where X_0^* , m_0 are as in Proposition 4.1).

Proposition 6.1. *For any $\rho > 0$ with $F(\rho) < 1$, and any $T > 0$, we have*

$$F(\rho \exp(-T)) \geq \frac{1}{2} \exp(-2T) \rho F'(\rho).$$

Proof. We apply Corollary 4.2 with $B = X_0^*$, and $\omega_0 = \frac{\pi}{2}$. From this corollary and Lemma 3.5, we obtain

$$\begin{aligned} m(Y(T, \frac{\pi}{2}, X_0^*)) &= \frac{1}{4} \exp(-2T) m_0(X_0^*) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log \frac{1 + \cos \omega}{1 - \cos \omega} \cos \omega \, d\omega \\ &= \frac{\pi}{2} \exp(-2T) m_0(X_0^*) \\ &= \frac{1}{2} \exp(-2T) \rho F'(\rho). \end{aligned}$$

To get the estimate of the proposition, it is now sufficient to recall that, by Proposition 3.4, any surface in $Y(T, \frac{\pi}{2}, X_0^*)$ has a systole $\leq \rho \exp(-T)$. \square

Corollary 6.2. *The function F is absolutely continuous, and its derivative satisfies*

$$F'(\rho) = O(\rho).$$

Moreover, defining

$$c(m) := \sup_{F(\rho) < 1} \frac{F'(\rho)}{\rho},$$

one has

$$\lim_{\rho \rightarrow 0} \frac{F(\rho)}{\rho^2} = \frac{1}{2} c(m)$$

and also

$$c(m) := \limsup_{\rho \rightarrow 0} \frac{F'(\rho)}{\rho}.$$

Proof. It follows from Proposition 6.1 that we have $F'(\rho) \leq 2\rho^{-1}F(\rho)$ for $F(\rho) < 1$. Recalling (cf. Remark 1.1) that $F(\rho) = O(\rho^2)$, we obtain that $F'(\rho) = O(\rho)$. In particular, F' is bounded.

If f is a continuous function on some interval $[\rho_0, \rho_1]$, having at each point a left-derivative bounded by C , then one has

$$|f(\rho_1) - f(\rho_0)| \leq C(\rho_1 - \rho_0)$$

and thus f is absolutely continuous. Indeed, it is sufficient to prove this inequality for any $C' > C$; but the supremum of $\rho \in [\rho_0, \rho_1]$ such that $|f(\rho) - f(\rho_0)| \leq C'(\rho - \rho_0)$ can only be equal to ρ_1 .

Applying this with $f = F$, we conclude that F is absolutely continuous.

Defining $c(m)$ as in the corollary, we get from Proposition 6.1 that

$$\liminf_{\rho \rightarrow 0} \frac{F(\rho)}{\rho^2} \geq \frac{1}{2}c(m).$$

As F is the integral of its left-derivative F' , we also have

$$\limsup_{\rho \rightarrow 0} \frac{F(\rho)}{\rho^2} \leq \frac{1}{2}c(m).$$

This proves that $\lim_{\rho \rightarrow 0} \rho^{-2}F(\rho) = \frac{1}{2}c(m)$. The last assertion of the corollary is then obvious. \square

Proof of Theorem 1.3. We have to prove that the m -measure of the set of $M \in \mathcal{C}$ which have at least two non-parallel saddle-connections of length $\leq \rho$ is $o(\rho^2)$ when ρ is small. Taking into account Remark 1.1, it is sufficient to prove that, for any $A > 1$, the set $\mathcal{C}(A, \rho)$ formed by the points $M \in \mathcal{C}$ with $\text{sys}(M) \leq \rho$ having another saddle-connection of length $\leq A \cdot \text{sys}(M)$, non parallel to the minimizing one, has m -measure $o(\rho^2)$ when ρ is small.

Fix some $A > 1$, and some $\eta > 0$.

We will prove that

$$(6.1) \quad m(\mathcal{C}(A, \rho)) < \eta\rho^2,$$

when ρ is sufficiently small.

We first choose ρ_0 sufficiently small to satisfy $F(\rho_0) < 1$ and

$$(6.2) \quad \rho_0^{-1}F'(\rho_0) > c(m) - \frac{\eta}{2};$$

$$(6.3) \quad \rho^{-2}F(\rho) < \frac{1}{2}(c(m) + \frac{\eta}{2}) \quad \text{for } 0 < \rho < \rho_0.$$

We use ρ_0 as the level of the systole function at which are defined X_0^* , m_0 in Section 4.

Let $\omega_0 = \omega_0(\eta) > 0$ be small enough to have

$$(6.4) \quad m_0(X_0^*) \int_{-\omega_0}^{\omega_0} \log \frac{1 + \cos \omega}{1 - \cos \omega} \cos \omega \, d\omega < \eta \rho_0^2.$$

Next, consider $T > 0$ and θ such that $\exp(-2T) \sin \omega_0 < |\sin 2\theta| < \exp(-2T)$. An easy calculation shows that there exists $K = K(\omega_0)$, **independent of** T , such that, for all $t \in J(T, \theta)$, the norm of the image of the vertical basis vector e_2 under $g_t R_\theta$ is $\leq K \exp(-t)$.

Thus, if v is any vector with $\|v\| \geq AK$, we will have, for such θ and t

$$(6.5) \quad \|g_t R_\theta v\| \geq AK \exp(-t) \geq A \|g_t R_\theta e_2\|.$$

Let $M \in X_0^*$. Denote by $\bar{\theta}(M)$ the minimal angle between a length-minimizing saddle-connection and another non-parallel saddle connection of length $\leq AK \rho_0$. If no such short non-parallel saddle connection exists, set $\bar{\theta}(M) = \frac{\pi}{2}$.

As $\bar{\theta}$ is everywhere positive on X_0^* , there exists $\bar{\theta}_0$ such that the set $B(\bar{\theta}_0) := \{M \in X_0^*, \bar{\theta}(M) < \bar{\theta}_0\}$ satisfies

$$(6.6) \quad \frac{\pi}{2} m_0(B(\bar{\theta}_0)) < \frac{\eta}{4} \rho_0^2.$$

On the other hand, if T is sufficiently large (say $T \geq T_0 = T_0(A, \bar{\theta}_0)$), one has, for $|\sin 2\theta| < \exp(-2T)$, $\frac{\pi}{2} > |\theta'| > \bar{\theta}_0$, $t \in J(T, \theta)$

$$(6.7) \quad \|g_t R_{\theta+\theta'} e_2\| \geq A \|g_t R_\theta e_2\|.$$

We claim that (6.1) holds for $\rho \leq \rho_0 \exp(-T_0)$. Indeed, writing $\rho = \rho_0 \exp(-T)$ with $T \geq T_0$:

- The measure of elements in $\{\text{sys}(M) \leq \rho\}$ which are **not** in $Y(T, \frac{\pi}{2}, X_0^*)$ is $< \frac{\eta}{2} \rho^2$. Indeed, from (6.2), (6.3) and Corollary 4.2, we have

$$F(\rho) < \frac{1}{2} (c(m) + \frac{\eta}{2}) \rho^2$$

and

$$m(Y(T, \frac{\pi}{2}, X_0^*)) = \frac{1}{2} \exp(-2T) \rho_0 F'(\rho_0) > \frac{1}{2} \rho^2 (c(m) - \frac{\eta}{2}).$$

- For $M \in Y(T, \frac{\pi}{2}, X_0^*)$, write $M = g_t R_\theta M_0$, with $M_0 \in X_0^*$, $|\sin 2\theta| < \exp(-2T)$, $t \in J(T, \theta)$. From (6.4) and Corollary 4.2, we get

$$m(Y(T, \omega_0, X_0^*)) < \frac{1}{4} \eta \rho^2.$$

Similarly, from (6.6) and Corollary 4.2, we get

$$m(Y(T, \frac{\pi}{2}, B(\bar{\theta}_0))) < \frac{1}{4} \eta \rho^2.$$

- Assume that $M = g_t R_\theta M_0$, with $\bar{\theta}(M_0) \geq \bar{\theta}_0$, $|\sin 2\theta| \geq \exp(-2T) \sin \omega_0$. Assume also that $T \geq T_0$. Then the point M does not belong to $\mathcal{C}(A, \rho)$. This follows from (6.5) for the saddle connections of M_0 of length $> AK \rho_0$, and from (6.7) for the saddle connections of M_0 of length $\leq AK \rho_0$.

This concludes the proof of (6.1) and also of the theorem. \square

REFERENCES

- [1] A. Avila and M. Viana, *Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture*, Acta Math., v. 198 (2007), 1–56.
- [2] I. Bouw and M. Möller, *Teichmüller curves, triangle groups, and Lyapunov exponents*, Ann. of Math., v. 172 (2010), 139–185.
- [3] V. Delecroix, P. Hubert and S. Lelièvre, *Diffusion for the periodic wind-tree model*, preprint, 2011, arXiv:1107.1810.
- [4] A. Eskin, M. Kontsevich and A. Zorich, *Lyapunov spectrum of square-tiled cyclic covers*, J. Mod. Dyn. v. 5 (2011), 319–353.
- [5] A. Eskin, M. Kontsevich and A. Zorich, *Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmüller geodesic flow*, preprint, 2011, arXiv:1112.5872.
- [6] A. Eskin and H. Masur, *Asymptotic formulas on flat surfaces*, Ergod. Th. and Dynam. Sys., v. 21 (2001), 443–478.
- [7] A. Eskin, H. Masur and A. Zorich, *Moduli spaces of Abelian differentials: the principal boundary, counting problems, and the Siegel-Veech constants*, Publications Mathématiques de l’IHÉS, v. 97 (2003), 61–179.
- [8] G. Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*, Ann. of Math., v.155 (2002), no. 1, 1–103.
- [9] G. Forni, *A geometric criterion for the non-uniform hyperbolicity of the Kontsevich–Zorich cocycle*, J. Mod. Dyn., v. 5 (2011), 355–395.
- [10] J. Grivaux and P. Hubert, *Les exposants de Liapounoff du flot de Teichmüller (d’après Eskin-Kontsevich-Zorich)*, Séminaire Bourbaki 2012–2013, n. 1060.
- [11] M. Kontsevich, *Lyapunov exponents and Hodge theory*, in ‘The mathematical beauty of physics’, Saclay, 1996. Adv. Ser. Math. Phys. v. 24, 318–332, World Scientific, River Edge, NJ, 1997.
- [12] H. Masur, *Interval exchange transformations and measured foliations*, Ann. of Math., v. 115 (1982), 169–200.
- [13] H. Masur, *The growth rate of trajectories of a quadratic differential*, Ergod. Th. Dynam. Sys., v. 10 (1990), 151–176.
- [14] H. Masur and J. Smillie, *Hausdorff dimension of sets of nonergodic foliations*, Ann. of Math., v. 134 (1991), 445–543.
- [15] C. Matheus, M. Möller and J.-C. Yoccoz, *A criterion for the simplicity of the Lyapunov exponents of square-tiled surfaces*, in preparation.
- [16] D.-M. Nguyen, *Volumes of the sets of translation surfaces with small saddle connections*, preprint, 2012, arXiv:1211.7314.
- [17] V. Rokhlin, *On the fundamental ideas of measure theory*, Mat. Sbornik N.S. 25(67), (1949). 107–150.
- [18] W. Veech, *Gauss measures for transformations on the space of interval exchange maps*, Ann. of Math., v. 115 (1982), 201–242.
- [19] ———, *Siegel measures*, Annals of Math., v. 148 (1998), 895–944.
- [20] J.-C. Yoccoz, *Interval exchange maps and translation surfaces*, Clay Math. Inst. Summer School on Homogenous Flows, Moduli Spaces and Arithmetic, Pisa (2007), available at http://www.college-de-france.fr/media/equ_dif/UPL15305_PisaLecturesJCY2007.pdf
- [21] L.-S. Young, *Ergodic theory of differentiable dynamical systems*, Real and complex dynamical systems (1993), 293–336, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 464, Kluwer Acad. Publ., Dordrecht, 1995.
- [22] A. Zorich, *Asymptotic flag of an orientable measured foliation on a surface*, in ‘Geometric Study of Foliations’, World Scientific (1994), 479–498.
- [23] ———, *Deviation for interval exchange transformations*, Ergod. Th. and Dynam. Sys., v. 17 (1997), 1477–1499.
- [24] ———, *Flat Surfaces*, Frontiers in number theory, physics, and geometry, Springer (2006), 437–583.

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